

Non-Linear Flow of a Viscoelastic Fluid between Two Eccentric Spheres

Mostafa Y. El-Bakry

Physics Department, Faculty of Science, Tabuk University
oelbakre@yahoo.com

Abstract: The study of the Non-Linear flow of a viscoelastic second-order fluid in the annular region between two eccentric spheres is investigated. This flow is created by considering the inner sphere to rotate with angular velocity Ω about the axis passing through the centers of two spheres, the z-axis, while the outer sphere is kept at rest using the Bispherical coordinates (α, β, ϕ) . The equations of motions of second order fluid using retarded motion approximation including inertia term are formulated and the particular solution of the modified planar second-order equation of motion which including inertia term is obtained. A modified second-order stream function ψ_2 , due to the effect of the inertia, which describes the planar secondary flow field has been properly obtained

Keywords: Non-Linear flow- Viscoelastic fluid-Bispherical coordinates - Biharmonic vector equation -Secondary flow- Stream function

1. Introduction

The Construction of Rheometers are based on solution of a boundary value problem. The theoretical and experimental studies concerning the flow of viscous or viscoelastic fluids in the annular narrow gaps between two rotating bodies [1-3] are very interesting boundary value problems in rheology. These problems represent the keystone in the high developing today industries and technology such as the flow in rotation turbo machinery, in journal-bearing lubrication, socket joints, petroleum and so on. One of these problems, for two concentric spheres the studies are carried out, say by Wimmer and Yamaguchi et. al. A large number of theoretical and experimental works are done on the viscous flow between two eccentric spheres; Majumdar, Munson, Menguturk and Munson. Abu-El Hassan et .al. studied the flow of viscoelastic fluid between two eccentric spheres and calculated the velocity for first order only. Recently, Elbakry et. al. studied forces and torques of eccentric sphere model using Genetic programming and Neural network.

The present work is concerned with the solution of this boundary value problem, including the effect of inertia. The velocity field up to a second order is being a superposition of a first order primary flow distributed uniformly around the axis of rotation and a secondary flow which is every where perpendicular to the streamlines of the primary flow. The streamlines of the second-order velocity is changed due the effect of the inertia.

2. Formulation of the problem

A viscoelastic second-order fluid is assumed to perform steady and isochoric motion in the annular region between two eccentric spheres . This flow is created by considering the inner sphere to rotate with angular velocity Ω about the axis connecting their centers, the z-axis, while the outer sphere is kept at rest.

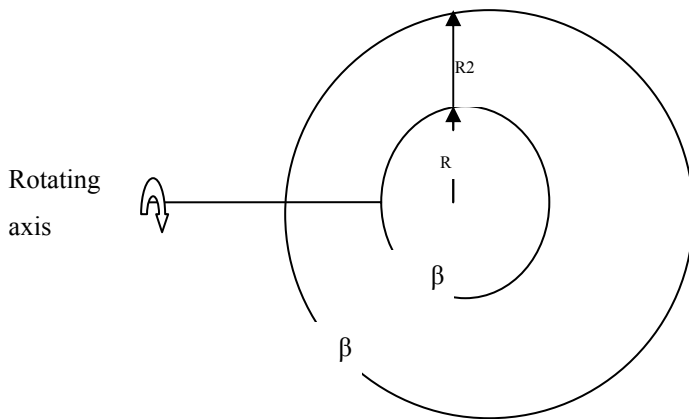


Fig.(1) Configuration of the two eccentric spheres

Using the Bispherical coordinates (α, β, ϕ) the inner sphere is the surface defined by $\beta = \beta_1$, while the outer one is defined by $\beta = \beta_2$. The radii of the inner and the outer spheres are then given by; [14],

$$R_1 = \frac{c}{sh \beta_1} \quad \text{and} \quad R_2 = \frac{c}{sh \beta_2}$$

Where c is a parameter related to the scale factors of the coordinates by the following relations:

$$h_\alpha = h_\beta = h = \frac{c}{\cosh \beta - \cos \alpha}$$

$$h_\phi = h \sin \alpha$$

Symmetry about the z-axis implies that the velocity field $\underline{\dot{x}}$ is independent of the coordinate ϕ . Hence, the velocity field can be stated in the form

$$\underline{\dot{x}}(\alpha, \beta) = u(\alpha, \beta)\hat{\alpha} + v(\alpha, \beta)\hat{\beta} + W(\alpha, \beta)\hat{\phi} \tag{1}$$

where the components $u(\alpha, \beta)$, $v(\alpha, \beta)$ and $W(\alpha, \beta)$ are only functions of the coordinates α and β .

The non-slip at the boundaries β_1 and β_2 imposes the boundary conditions

$$\left. \begin{aligned} \dot{\underline{x}}|_{\beta=\beta_1} &= \hat{\varphi} \underline{\Omega} h_\varphi(\alpha, \beta_1) \\ \dot{\underline{x}}|_{\beta=\beta_2} &= 0. \end{aligned} \right\} \tag{2}$$

The equation of continuity ,

$$\nabla \cdot \dot{\underline{x}} = \frac{1}{h^3 \sin \alpha} \left[\frac{\partial}{\partial \alpha} (h^2 \sin \alpha u) + \frac{\partial}{\partial \beta} (h^2 \sin \alpha v) \right] = 0 .$$

is satisfied identically if u and v are derivable from a stream function ψ by the expression

$$u = \frac{-1}{h^2 \sin \alpha} \frac{\partial \psi}{\partial \beta} \quad \text{and} \quad v = \frac{1}{h^2 \sin \alpha} \frac{\partial \psi}{\partial \alpha} \tag{3a}$$

Or in the compact form we can write

$$\underline{U} = u(\alpha, \beta) \hat{\alpha} + v(\alpha, \beta) \hat{\beta} = \nabla_{\mathbf{x}} \left(\hat{\varphi} \frac{c\psi}{h_\varphi} \right) . \tag{3b}$$

The Cauchy dynamical equation of motion for a stationary flow is being

$$\nabla \cdot \underline{T}(\alpha, \beta) = \nabla \cdot \underline{T}_E(\alpha, \beta) - \nabla \chi(\alpha, \beta) = \rho \dot{\underline{x}}(\alpha, \beta) \cdot \nabla \dot{\underline{x}}(\alpha, \beta) , \tag{4}$$

where \underline{T} is the stress tensor, \underline{T}_E is the extra stress tensor, χ is the hydrostatic pressure function and ρ is the density of the fluid .

The problem is dealt within the frame of the retarded motion approximation[17,18]. Within this frame, the following simplifications can be used :

(i) The constitutive equation for a second-order fluid is stated as follow

$$\underline{T}_E = \mu \underline{A}_1 + \alpha_1 \underline{A}_2 + \alpha_2 \underline{A}_1^2 , \tag{5}$$

where μ is the coefficient of viscosity, α_1 and α_2 are two second- order material coefficients related to the normal stress differences. The tensors \underline{A}_1 and \underline{A}_2 are the first two Rivlin-Ericksen tensors defined for steady state flow , by

$$\underline{A}_1 = \nabla \dot{\underline{x}} + (\nabla \dot{\underline{x}})^T , \tag{6a}$$

$$\underline{A}_2 = \dot{\underline{x}} \cdot \nabla \underline{A}_1 + \nabla \dot{\underline{x}} \cdot \underline{A}_1 + (\nabla \dot{\underline{x}} \cdot \underline{A}_1)^T . \tag{6b}$$

(ii)The functions W, ψ and χ can be expanded into power series about the value $\Omega = 0$, hence

$$\begin{aligned} W &= \sum_{k=1}^{n-1} \Omega^k W_k + 0(\Omega^n) \\ \psi &= \sum_{k=1}^{n-1} \Omega^k \psi_k + 0(\Omega^n) \\ \chi &= \sum_{k=1}^{n-1} \Omega^k \chi_k + 0(\Omega^n) \end{aligned} \tag{7}$$

The method of perturbation can be outlined in the following steps:

- (a) Substitution from (7) into (6a) and (6b) and further into (5) gives an expression for the extra-stress tensor into powers of Ω .
- (b) The result can be substituted into equation (4).
- (c) Equating the coefficients of equal powers of Ω produces a set of successive partial differential equations for the determination of the velocity components in successive order.

3. First-order approximation

Within this order of approximation the extra-stress tensor \underline{T}_E includes only \underline{A}_1 , up to $O(\Omega^2)$, is given by

$$\underline{T}_E = \mu \underline{A}_1 = \mu \Omega [\nabla \underline{U}_1 + (\nabla \underline{U}_1)^T + \nabla(W_1 \hat{\phi}) + (\nabla(W_1 \hat{\phi}))^T] + O(\Omega^2), \tag{8}$$

The first-order axial velocity satisfies the harmonic equation ,

$$\nabla^2(W_1 \hat{\phi}) = 0. \tag{9}$$

with the boundary conditions ,

$$W_1(\alpha, \beta_1) = h_\varphi(\alpha, \beta_1), W_1(\alpha, \beta_2) = 0. \tag{10}$$

This boundary value problem has the solution

$$W_1(\alpha, \beta) = 2^{\frac{3}{2}} C (\cosh \beta - \cos \alpha)^{1/2} \sum_{n=1}^{\infty} \frac{\sinh(n+1/2)(\beta - \beta_2)}{\sinh(n+1/2)\delta e^{(n+1/2)\beta_1}} P_n^1(\cos \alpha), \tag{11}$$

Where, P_n^1 is the associated Legendre polynomials of order n and degree one and $\delta = \beta_1 - \beta_2$.

On the other hand, the stream function ψ_1 satisfies the boundary value problem,

$$\nabla^4 \left(\frac{c \psi_1(\alpha, \beta)}{h_\varphi} \hat{\phi} \right) = 0. \tag{12}$$

with the boundary conditions

$$\psi_1(\alpha, \beta) = \partial_\beta \psi_1(\alpha, \beta) = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \text{ at } \beta = \begin{Bmatrix} \beta_1 \\ \beta_2 \end{Bmatrix}. \tag{13}$$

It can be easily shown that the only solution for the boundary value problem defined by equations (12) and (13) is the trivial solution $\psi_1=0$.

Up to the first order , the velocity field reduces to,

$$\underline{\dot{x}} = \Omega W_1(\alpha, \beta) \hat{\phi} + O(\Omega^2) \tag{14}$$

4. Second-order approximation ignoring inertia

Up to this order of approximation the Cauchy dynamical equation of motion which is given by Eq.(4) ignoring inertia term $\rho \underline{\dot{x}}(\alpha, \beta) \cdot \nabla \underline{\dot{x}}(\alpha, \beta)$ becomes

$$\nabla \bullet [\mu \underline{\underline{A}}_1 + \alpha_1 \underline{\underline{A}}_2 + \alpha_2 \underline{\underline{A}}_1^2] - \nabla \chi_2 = 0, \tag{15}$$

where χ_2 is the second order pressure term .During our calculations, we have to notice that from equations (6a) and (9) up to the $O(\Omega_1^2)$

$$\nabla \bullet \underline{\underline{A}}_1 = 0 . \tag{16}$$

Consequently,

$$\nabla \bullet \underline{\underline{A}}_2 = \frac{1}{2} \nabla (\underline{\underline{A}}_1 : \underline{\underline{A}}_1) + 2(\nabla \underline{\underline{x}} : \nabla \nabla \underline{\underline{x}}). \tag{17}$$

Similarly,

$$\nabla \bullet \underline{\underline{A}}_1^2 = \frac{1}{4} \nabla (\underline{\underline{A}}_1 : \underline{\underline{A}}_1) + 2(\nabla \underline{\underline{x}} : \nabla \underline{\underline{x}}) \tag{18}$$

Substituting equations (16), (17) and (18) into equation (15) we get

$$\begin{aligned} \mu \left[\nabla^2 (W_2 \hat{\phi}) + \nabla^2 \left[\nabla_x \left(\frac{c \psi_2(\alpha, \beta) \hat{\phi}}{h_\phi} \right) \right] \right] + 2(\alpha_1 + \alpha_2)(\nabla \underline{\underline{x}} : \nabla \nabla \underline{\underline{x}}) \\ + \frac{1}{4}(\alpha_2 + 2\alpha_1)\nabla (\underline{\underline{A}}_1 : \underline{\underline{A}}_1) - \nabla \chi_2(\alpha, \beta) = 0. \end{aligned} \tag{19}$$

Applying the decomposition performed in the first order approximation to this equation, it decomposes into two equations with the relevant boundary conditions. The first boundary value problem determine the ϕ -component of velocity $W_2(\alpha, \beta)$, which has the form

$$\nabla^2 (W_2 \hat{\phi}) = 0, \tag{20}$$

with the boundary conditions

$$W_2(\alpha, \beta) = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \text{ at } \beta = \begin{Bmatrix} \beta_1 \\ \beta_2 \end{Bmatrix}. \tag{21}$$

This boundary value problem has the identity solution

$$W_2(\alpha, \beta) = 0. \tag{22}$$

The second boundary value problem concerned the stream function $\psi_2(\alpha, \beta)$ is

$$\begin{aligned} \mu \nabla^2 \left[\nabla_x \left(\frac{c \psi_2(\alpha, \beta) \hat{\phi}}{h_\phi} \right) \right] + 2(\alpha_1 + \alpha_2)(\nabla \underline{\underline{x}} : \nabla \nabla \underline{\underline{x}}) \\ + \frac{1}{4}(\alpha_2 + 2\alpha_1)\nabla (\underline{\underline{A}}_1 : \underline{\underline{A}}_1) - \nabla \chi_2(\alpha, \beta) = 0. \end{aligned} \tag{23}$$

Taking the curl of the last equation we get

$$\nabla^4 \left(\frac{c \psi_2 \hat{\phi}}{h_\phi} \right) = \frac{2(\alpha_2 + \alpha_1)}{\mu} \nabla_x (\nabla \underline{\underline{x}} : \nabla \nabla \underline{\underline{x}}). \tag{24}$$

Using equation(11) and equation(14) in equation(24),it can be shown that

$$\nabla^4 \left(\frac{c \psi_2 \hat{\phi}}{h_\phi} \right) = \frac{144c^2 (\alpha_1 + \alpha_2) \sin \alpha}{\mu sh^2 (\frac{3}{2} \delta) h^4} \left(\begin{array}{l} 2 \cos \alpha \left[\frac{h}{c} (ch\beta F_1(\beta) + sh^2 \beta F_2(\beta)) + F_1(\beta) \right] \\ - \frac{h}{c} \sin^2 \alpha [2F_1(\beta) + \frac{3}{4} sh\beta] - F_2(\beta) \end{array} \right) \hat{\phi}, \tag{25}$$

Where, $F_1(\beta) = sh\beta sh^2 \frac{3}{2} (\beta - \beta_2)$

and $F_2(\beta) = sh3(\beta - \beta_2)$

with the boundary conditions

$$\psi_2(\alpha, \beta) = \partial_\beta \psi_2(\alpha, \beta) = \left\{ \begin{array}{l} 0 \\ 0 \end{array} \right\} \quad \text{at} \quad \beta = \left\{ \begin{array}{l} \beta_1 \\ \beta_2 \end{array} \right\} \tag{26}$$

It is obvious that there exist a non-zero unique solution for $\psi_2(\alpha, \beta)$ which describes a secondary flow field \underline{U}_2 superimposed onto the primary flow $W_1(\alpha, \beta)$. The solution of Eq.(25) is performed in [15]. The final form of the second-order stream function Ψ_2 is given by,

$$\psi_2(\alpha, \beta) = \frac{36h^2c(\alpha_1 + \alpha_2) \sin^2 \alpha}{\mu \pi sh^2 (\frac{3}{2} \delta)} \left\{ \begin{array}{l} S1(\beta) + S2(\beta) + q_1 \frac{h^3 sh^3 \beta}{c^3} \\ + q_2 \frac{h}{c} (ch\beta + \cos \alpha) + q_3 \frac{hsh\beta}{c} + q_4 \end{array} \right\}, \tag{27}$$

where the functions, $S1(\beta), S2(\beta)$ are given by,

$$S1 = \ln \left(\frac{ch\beta + 1}{ch\beta - 1} \right) \left\{ f1(\beta) - e^\beta \cos \alpha [f2(\beta_0)]_{\beta_1}^{\beta_1} - e^{-\beta} \cos \alpha [f3(\beta_0)]_{\beta_2}^{\beta_2} \right\}, \tag{28a}$$

$$S2 = e^\beta [\cos \alpha f4(\beta_0) + \sin \alpha f5(\beta_0)]_{\beta_1}^{\beta_1} + e^{-\beta} [\cos \alpha f6(\beta_0) + \sin \alpha f7(\beta_0)]_{\beta_2}^{\beta_2} + f8(\beta), \tag{28b}$$

With,

$$\begin{aligned} f1(\beta) &= 10^{-3} \pi \left[13.6\beta ch(3\beta_2 - 8\beta) - 50\beta ch(3\beta_2 - 6\beta) + 39\beta ch(3\beta_2 - 4\beta) + 14.5\beta sh(3\beta_2) sh(2\beta) + \right. \\ &\quad \left. 1.7sh(3\beta_2 - 8\beta) - 8.2sh(3\beta_2 - 6\beta) + 9sh(3\beta_2 - 4\beta) - 7.2sh(3\beta_2) ch(2\beta) + 504(sh\beta - \beta ch\beta) \right. \\ &\quad \left. 74\beta^2 sh(3\beta_2) + 112\beta ch3\beta - 23\beta ch5\beta + 37sh3\beta + 4.7sh5\beta \right] \\ f2(\beta) &= 10^{-2} [6.3e^\beta - 1.68e^{3\beta} + .42e^{5\beta} - .2e^{8\beta-3\beta_2} - .18e^{2\beta+3\beta_2} + .42e^{2\beta} sh3\beta_2 + .075e^{4\beta-3\beta_2} \\ &\quad - .04e^{6\beta+3\beta_2} - .48\beta sh3\beta_2] \\ f3(\beta) &= 10^{-2} [-1.63e^{3\beta} + e^{5\beta} - .06e^{7\beta} - .96e^{2\beta} ch3\beta_2 + .09e^{4\beta+3\beta_2} - .05e^{6\beta-3\beta_2} - .39e^{8\beta-3\beta_2} \\ &\quad + .166e^{10\beta-3\beta_2} + 1.92\beta ch3\beta_2] \\ f4(\beta) &= 10^{-2} [-15\beta + 6.8e^{2\beta} - 1.5e^{4\beta} - 1.5e^\beta sh3\beta_2 + 2.33sh3(\beta - \beta_2) + 3.4e^{3\beta-3\beta_2} - 1.6e^{5\beta-3\beta_2} + .77e^{7\beta-3\beta_2}] \\ f5(\beta) &= \frac{\pi}{16} [3.3e^\beta - .44e^{3\beta} + .044e^{5\beta} - .03e^{2\beta+3\beta_2} - .23e^{4\beta-3\beta_2} + .06e^{6\beta-3\beta_2} - .008e^{8\beta-3\beta_2} + .76\beta ch3\beta_2] \\ f6(\beta) &= 10^{-2} [20\beta - 7.3e^{2\beta} + 4e^{4\beta} - 1.25e^{6\beta} + 1.45e^{5\beta-3\beta_2} - 1.36e^{7\beta-3\beta_2} + 2.2e^{9\beta-3\beta_2} \\ &\quad e^{3\beta} (-3.6 + .85 sh3\beta_2) + 1.2e^\beta (e^{3\beta_2} + 10 sh3\beta_2) - 2.8e^{5\beta+\frac{3}{2}\beta_2} + .9e^{2\beta} sh \frac{3}{2} (\beta - \beta_2)] \\ f7(\beta) &= \frac{\pi}{16} [-.5e^\beta + 1.1e^{3\beta} - .26e^{5\beta} + .03e^{7\beta} + e^{-3\beta_2} (-.3e^{4\beta} - .15e^{6\beta} + .05e^{8\beta} - .006e^{10\beta}) \\ &\quad + e^{3\beta_2} (-.36\beta - .63e^{-2\beta} + .8e^{2\beta} - .11e^{4\beta})] \end{aligned}$$

$$f_8(\beta) = \pi \left\{ \beta \left[.028 \operatorname{ch}(3\beta_2 - 7\beta) - .11 \operatorname{ch}(3\beta_2 - 5\beta) - .11 \operatorname{ch}(3\beta_2 - 3\beta) - .14 \operatorname{ch}(3\beta_2 - \beta) \right. \right. \\ \left. \left. + .03 \operatorname{ch}(3\beta_2 + 3\beta) + .38 \operatorname{ch} 2\beta - .062 \operatorname{ch} 4\beta \right] - \operatorname{sh}(3\beta_2 - 7\beta) - .015 \operatorname{sh}(3\beta_2 - 5\beta) \right. \\ \left. + .028 \operatorname{sh}(3\beta_2 - 3\beta) - .1 \operatorname{sh}(3\beta_2 - \beta) - .046 \operatorname{sh}(3\beta_2 + \beta) - .12 \operatorname{sh} 2\beta + .11 \operatorname{sh} 4\beta + \right. \\ \left. 10^{-2} [-4 \operatorname{sh} 3\beta_2 \ln(\operatorname{sh} \beta) + \beta^2 (34 - 4 \operatorname{ch} 3\beta_2) - 8\beta^3 \operatorname{sh} 3\beta_2] \right\}$$

After applying the boundary conditions the constants q_1, q_2, q_3 and q_4 are given by

$$q_1 = \frac{\omega_1 \omega_2 - \omega_3 \omega_4}{\omega_4 \omega_5 - \omega_1 \omega_6}, \tag{29a}$$

$$q_2 = \frac{-1}{\omega_1} [\omega_3 + \omega_5 q_1] \tag{29b}$$

$$q_3 = \frac{-1}{g_1(\beta_1) - g_1(\beta_2)} \{E(\beta_1) - E(\beta_2) + q_2 [g_2(\beta_1) - g_2(\beta_2)] + q_1 [g_1^3(\beta_1) - g_1^3(\beta_2)]\} \tag{29c}$$

$$q_4 = -[E(\beta_1) + q_3 g_1(\beta_1) q_2 g_2(\beta_1)] + q_1 g_1^3(\beta_1), \tag{29d}$$

where ,

$$g_1(\beta) = h(\beta) \operatorname{sh} \beta$$

$$g_2(\beta) = h(\beta) (\operatorname{ch} \beta + \cos \alpha),$$

$$g_3(\beta) = h^2(\beta) (1 - \operatorname{ch} \beta \cos \alpha)$$

$$g_4(\alpha) = -2g_1^3(\beta_2) + 2g_1(\beta_1)g_1(\beta_2) + h^2(\beta_2)(1 - \operatorname{ch} \beta_2 \cos \alpha)$$

$$E(\beta) = \frac{\mu \pi \operatorname{sh}^2(\frac{3}{2} \delta) [S_1(\beta) + S_2(\beta)]}{36(\alpha_1 + \alpha_2) h^2 c \sin^2 \alpha}$$

$$\omega_1 = -2g_1(\beta_1) h(\beta_1) \cos \alpha - \frac{[g_2(\beta_1) - g_2(\beta_2)]}{[g_1(\beta_1) - g_1(\beta_2)]} g_3(\beta_1)$$

$$\omega_2 = E_{,\beta}(\beta_2) + [E(\beta_1) - E(\beta_2)] \left(2g_1(\beta_2) - \frac{g_4(\beta_2)}{[g_1(\beta_1) - g_1(\beta_2)]} \right)$$

$$\omega_3 = E_{,\beta}(\beta_1) - \frac{[E(\beta_1) - E(\beta_2)] g_3(\beta_1)}{[g_1(\beta_1) - g_1(\beta_2)]}$$

$$\omega_4 = -2g_1(\beta_2) h(\beta_2) \cos \alpha + [g_2(\beta_1) - g_2(\beta_2)] \left(2g_1(\beta_2) - \frac{g_4(\alpha)}{[g_1(\beta_1) - g_1(\beta_2)]} \right)$$

$$\omega_5 = g_3(\beta_1) \left(2g_1^2(\beta_1) - g_1^2(\beta_2) - g_1(\beta_1)g_1(\beta_2) \right)$$

$$\omega_6 = -2g_1^4(\beta_2) + 2g_1^3(\beta_1)g_1(\beta_2) + 3g_1^2(\beta_2)h^2(\beta_2)(1 - \operatorname{ch} \beta_2 \cos \alpha) - \frac{(g_1^3(\beta_1) - g_1^3(\beta_2))}{g_1(\beta_1) - g_1(\beta_2)}$$

5. Second-order approximation including inertia

The inertial term in the Cauchy dynamical equation of motion for a stationary flow, Eq.(4), becomes

$$\begin{aligned} \rho \underline{\dot{x}}(\alpha, \beta) \cdot \nabla \underline{\dot{x}}(\alpha, \beta) &= \rho w_1 \hat{\phi} \cdot \frac{\hat{\phi}}{h \sin \alpha} (w_1 \hat{\phi})_{,\varphi} = \rho \frac{w_1^2}{h \sin \alpha} (\hat{\phi})_{,\varphi} \\ &= \rho w_1^2 \left(\frac{ch\beta \cos \alpha - 1}{c \sin \alpha} \hat{\alpha} + \frac{sh\beta}{c} \hat{\beta} \right) \end{aligned} \tag{30}$$

therefore, up to this order of approximation Eq.(4), becomes

$$\nabla \bullet \left[\mu \underline{\underline{A}}_1 + \alpha_1 \underline{\underline{A}}_2 + \alpha_2 \underline{\underline{A}}_1^2 \right] - \nabla \chi_2 = \rho w_1^2 \left(\frac{ch\beta \cos \alpha - 1}{c \sin \alpha} \hat{\alpha} + \frac{sh\beta}{c} \hat{\beta} \right) \tag{31}$$

Substituting equations (16),(17), (18) and (30) into equation (31) we get

$$\begin{aligned} \mu \left[\nabla^2 (W_2 \hat{\phi}) + \nabla^2 \left[\nabla x \left(\frac{c \psi_2(\alpha, \beta) \hat{\phi}}{h_\varphi} \right) \right] \right] + 2(\alpha_1 + \alpha_2) (\nabla \underline{\dot{x}} : \nabla \nabla \underline{\dot{x}}) \\ + \frac{1}{4} (\alpha_2 + 2\alpha_1) \nabla (\underline{\underline{A}}_1 : \underline{\underline{A}}_1) - \nabla \chi_2(\alpha, \beta) = \rho w_1^2 \left(\frac{ch\beta \cos \alpha - 1}{c \sin \alpha} \hat{\alpha} + \frac{sh\beta}{c} \hat{\beta} \right). \end{aligned} \tag{32}$$

Applying the last decomposition performed in the second order approximation ignoring inertia to this equation which inertial term is considered, it decomposes into the following two equations with the relevant boundary conditions. The first boundary value problem is

$$\nabla^2 (W_2 \hat{\phi}) = 0, \tag{33}$$

with the boundary conditions

$$W_2(\alpha, \beta) = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \text{ at } \beta = \begin{Bmatrix} \beta_1 \\ \beta_2 \end{Bmatrix}. \tag{34}$$

This boundary value problem has the identity solution

$$W_2(\alpha, \beta) = 0. \tag{35}$$

The second boundary value problem concerned with the determination of the modified stream function $\psi_2(\alpha, \beta)$ is stated as follows

$$\begin{aligned} \mu \nabla^2 \left[\nabla x \left(\frac{c \psi_2(\alpha, \beta) \hat{\phi}}{h_\varphi} \right) \right] + 2(\alpha_1 + \alpha_2) (\nabla \underline{\dot{x}} : \nabla \nabla \underline{\dot{x}}) \\ + \frac{1}{4} (\alpha_2 + 2\alpha_1) \nabla (\underline{\underline{A}}_1 : \underline{\underline{A}}_1) - \nabla \chi_2(\alpha, \beta) = \rho w_1^2 \left(\frac{ch\beta \cos \alpha - 1}{c \sin \alpha} \hat{\alpha} + \frac{sh\beta}{c} \hat{\beta} \right). \end{aligned} \tag{36}$$

Taking the curl of the last equation we get

$$\nabla^4 \left(\frac{c \psi_2 \hat{\phi}}{h_\varphi} \right) = \frac{2(\alpha_2 + \alpha_1)}{\mu} \nabla x (\nabla \underline{\dot{x}} : \nabla \nabla \underline{\dot{x}}) + \frac{1}{\mu} \nabla x \left[\rho w_1^2 \left(\frac{ch\beta \cos \alpha - 1}{c \sin \alpha} \hat{\alpha} + \frac{sh\beta}{c} \hat{\beta} \right) \right]. \tag{37}$$

Using equation (11) and equation (14) in equation (37), we get

$$\nabla x \left[\rho \omega_1^2 \left(\frac{ch\beta \cos \alpha - 1}{c \sin \alpha} \hat{\alpha} + \frac{sh\beta}{c} \hat{\beta} \right) \right] = \hat{\phi} \left[\frac{3a^2 c^2 \rho}{h^2} \left([F_1(\beta) + F_2(\beta) \frac{ch\beta}{2}] \sin \alpha \cos \alpha - \frac{F_2(\beta)}{2} \sin \alpha \right) \right], \quad (38)$$

and

$$\nabla x (\nabla \underline{\dot{x}} : \nabla \nabla \underline{\dot{x}}) = \hat{\phi} \frac{72c^2 \sin \alpha}{sh^2(\frac{3}{2}\delta)h^4} \left(\begin{aligned} &2 \cos \alpha \left[\frac{h}{c} (ch\beta F_1(\beta) + sh^2 \beta F_2(\beta)) + F_1(\beta) \right] \\ &-\frac{h}{c} \sin^2 \alpha [2F_1(\beta) + \frac{3}{4} sh\beta] - F_2(\beta) \end{aligned} \right) \quad (39)$$

Where, $a = 2^{\frac{3}{2}} \exp(-\beta_1) sh \frac{3}{2} \delta$, $F_1(\beta) = sh\beta sh^2 \frac{3}{2} (\beta - \beta_2)$

and $F_2(\beta) = sh3(\beta - \beta_2)$

Substitute from equations (38) and (39) into equation (37) we obtain the following biharmonic vector equation

$$\nabla^4 \left(\frac{c \psi_2 \hat{\phi}}{h_\varphi} \right) = \frac{144 c^2 (\alpha_1 + \alpha_2) \sin \alpha}{\mu sh^2(\frac{3}{2}\delta)h^4} \left(\begin{aligned} &2 \cos \alpha \left[\frac{h}{c} (ch\beta F_1(\beta) + sh^2 \beta F_2(\beta)) + F_1(\beta) \right] \\ &-\frac{h}{c} \sin^2 \alpha [2F_1(\beta) + \frac{3}{4} sh\beta] - F_2(\beta) \end{aligned} \right) \hat{\phi} \\ + \hat{\phi} \left[\frac{3a^2 c^2 \rho}{\mu h^2} \left([F_1(\beta) + F_2(\beta) \frac{ch\beta}{2}] \sin \alpha \cos \alpha - \frac{F_2(\beta)}{2} \sin \alpha \right) \right], \quad (40)$$

with the boundary conditions

$$\psi_2(\alpha, \beta) = \partial_\beta \psi_2(\alpha, \beta) = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \text{ at } \beta = \begin{Bmatrix} \beta_1 \\ \beta_2 \end{Bmatrix} \quad (41)$$

Equation (40) is an inhomogeneous biharmonic vector equation which can be decomposed into two equivalent scalar equations by expanding the vector $\hat{\phi}$ as follows :

$$\hat{\phi} = -\hat{x} \sin \varphi + \hat{y} \cos \varphi ;$$

thus , we get

$$\nabla^4 \left(\frac{c \psi_2}{h_\varphi} \begin{Bmatrix} \cos \varphi \\ \sin \varphi \end{Bmatrix} \right) = \left[\frac{144 c^2 (\alpha_1 + \alpha_2) \sin \alpha}{\mu sh^2(\frac{3}{2}\delta)h^4} \left(\begin{aligned} &2 \cos \alpha \left[\frac{h}{c} (ch\beta F_1(\beta) + sh^2 \beta F_2(\beta)) + F_1(\beta) \right] \\ &-\frac{h}{c} \sin^2 \alpha [2F_1(\beta) + \frac{3}{4} sh\beta] - F_2(\beta) \end{aligned} \right) \begin{Bmatrix} \cos \varphi \\ \sin \varphi \end{Bmatrix} \right] \\ + \left[\frac{3a^2 c^2 \rho}{\mu h^2} \left([F_1(\beta) + F_2(\beta) \frac{ch\beta}{2}] \sin \alpha \cos \alpha - \frac{F_2(\beta)}{2} \sin \alpha \right) \right] \begin{Bmatrix} \cos \varphi \\ \sin \varphi \end{Bmatrix}, \quad (42)$$

where the solution remains unchanged by taking either the $\cos \varphi$ or the $\sin \varphi$ function .

Renominating the function in Eq.(42) by ,

$$\psi^* = \frac{c \psi_2 \cos \varphi}{h_\varphi} ,$$

$$p^* = p^{*1} + p^{*2} , p^{*1} = p^1 \cos \varphi , p^{*2} = p^2 \cos \varphi ,$$

$$(43)$$

where,

$$p^1 = \frac{144c^2(\alpha_1 + \alpha_2) \sin \alpha}{\mu sh^2 (\frac{3}{2} \delta) h^4} \left(\begin{array}{l} 2 \cos \alpha \left[\frac{h}{c} (ch\beta F_1(\beta) + sh^2 \beta F_2(\beta)) + F_1(\beta) \right] \\ - \frac{h}{c} \sin^2 \alpha [2F_1(\beta) + \frac{3}{4} sh\beta] - F_2(\beta) \end{array} \right),$$

$$p^2 = \left[\frac{3a^2 c^2 \rho}{\mu h^2} \left([F_1(\beta) + F_2(\beta) \frac{ch\beta}{2}] \sin \alpha \cos \alpha - \frac{F_2(\beta)}{2} \sin \alpha \right) \right]$$

The function ψ^* is constructed, as stated before, as the sum of the two functions; namely,

$$\psi^* = \psi^{*p} + \psi^{*pin}, \tag{44}$$

ψ^{*p} is a solution of equation (43) includes \mathbf{p}^{*1} only which given by equation (27) and consists of a particular solution and the solution of the homogeneous biharmonic equation, $\nabla^4(\psi^{*h})=0$. ψ^{*pin} is a particular solution of equation (43) includes \mathbf{p}^{*2} only, inertia term, which obtained via a proper Green's function.

The Green's function which satisfies the problem under consideration is the solution of the inhomogeneous equation ; [11],

$$\nabla^4 G(\alpha, \beta, \varphi | \alpha_0, \beta_0, \varphi_0) = [-8\pi^2 h^{-3}] \delta(\alpha - \alpha_0) \delta(\beta - \beta_0) \delta(\varphi - \varphi_0), \tag{45}$$

where, $\delta(x - x_0)$ is the Dirac delta function .

A proper Green's function is of the form $[(R^2 / 4) \ln R]$; where, $R^2 = (x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2$

In terms of Bispherical coordinates (α, β, φ) this function takes the form

$$R^2 = 2hh_0 [ch(\beta - \beta_0) - \cos(\alpha - \alpha_0) - \sin \alpha \sin \alpha_0 (\cos(\varphi - \varphi_0) - 1)]$$

where , $h_0 = \frac{c}{ch\beta_0 - \cos \alpha_0}$

Using the following expansion of the logarithmic function which included in the Green's function

$$\ln(ch\beta - \cos \alpha) = \beta - 2 \sum_{n=1}^{\infty} \exp(-n\beta) \frac{\cos(n\alpha)}{n} - \ln 2$$

$$\ln(1-x) = - \sum_{n=1}^{\infty} \frac{x^n}{n}$$

Consequently, the Green's function $G(\alpha, \beta, \varphi | \alpha_0, \beta_0, \varphi_0)$ is given by

$$G(\alpha, \beta, \varphi | \alpha_0, \beta_0, \varphi_0) = hh_0 (G_1(\alpha, \varphi | \alpha_0, \varphi_0) + G_2(\alpha, \beta | \alpha_0, \beta_0) + G_3(\alpha, \beta, \varphi | \alpha_0, \beta_0, \varphi_0)), \tag{46}$$

$$G_1(\alpha, \varphi | \alpha_0, \varphi_0) = -\frac{1}{2} [\sin \alpha \sin \alpha_0 (\cos(\varphi - \varphi_0) - 1)]$$

$$G_2(\alpha, \beta | \alpha_0, \beta_0) = [ch(\beta - \beta_0) - \cos(\alpha - \alpha_0)] \mathbf{L}(\alpha, \beta | \alpha_0, \beta_0)$$

$$G_3(\alpha, \beta, \varphi | \alpha_0, \beta_0, \varphi_0) = -[\sin \alpha \sin \alpha_0 (\cos(\varphi - \varphi_0) - 1)] \mathbf{L}(\alpha, \beta | \alpha_0, \beta_0)$$

With

$$L(\alpha, \beta | \alpha_0, \beta_0) = (\ln 2 - \ln c) + \sum_{n=1}^{\infty} \exp(-n\beta) \frac{\cos(n\alpha)}{n} + \sum_{n=1}^{\infty} \exp(-n\beta_0) \frac{\cos(n\alpha_0)}{n} - \beta_{<} - \sum_{n=1}^{\infty} \exp(-n(\beta_{>} - \beta_{<})) \frac{\cos(n(\alpha - \alpha_0))}{n}$$

The Green's function theory implies that the particular solution, ψ^{*pin} is expressed by the integral,

$$\psi^{*pin}(\alpha, \beta, \varphi) = \frac{1}{8\pi^2} \int_0^{2\pi} d\varphi_0 \int_0^{\pi} d\alpha_0 \int_{\beta_1}^{\beta_2} p^{*2}(\alpha_0, \beta_0, \varphi_0) h_{\alpha}(\alpha_0, \beta_0) h_{\beta}(\alpha_0, \beta_0) h_{\varphi}(\alpha_0, \beta_0) G(\alpha, \beta, \varphi | \alpha_0, \beta_0, \varphi_0) d\beta_0 \quad (47)$$

The first three terms are included in the homogenous solution then the last equation becomes

$$\psi^{*pin}(\alpha, \beta, \varphi) = \frac{h \sin \alpha}{8\pi^2} \int_0^{2\pi} d\varphi_0 \int_0^{\pi} d\alpha_0 \int_{\beta_1}^{\beta_2} p^{*2}(\alpha_0, \beta_0) h^4(\alpha_0, \beta_0) \sin^2 \alpha_0 (1 - \cos(\varphi - \varphi_0)) [\beta_{<} + \sum_{n=1}^{\infty} \frac{e^{-n(\beta_{>} - \beta_{<})}}{n} \cos n(\alpha - \alpha_0)] d\beta_0 \quad (48)$$

performing the φ_0 -integration of the last equation we get ,

$$\psi^{*pin}(\alpha, \beta, \varphi) = \frac{h \sin \alpha \cos \varphi}{8\pi^2} \int_0^{\pi} d\alpha_0 \int_{\beta_1}^{\beta_2} p^{*2}(\alpha_0, \beta_0) h^4(\alpha_0, \beta_0) \sin^2 \alpha_0 [\beta_{<} + \sum_{n=1}^{\infty} \frac{e^{-n(\beta_{>} - \beta_{<})}}{n} \cos n(\alpha - \alpha_0)] d\beta_0 \quad (49)$$

Since the expression $\sum_{n=1}^{\infty} \frac{\exp(-n(\beta_{>} - \beta_{<}))}{n}$ is a fast convergence function, it can be approximated by its leading term. This approximation is necessary in order to keep the last integrations in a tractable limits. This step is tested numerically with the result that this truncation does not affect the final result seriously.

Substituting p^{*2} from equation (43) and taking the summation for $n=1$ then equation (49) becomes ,

$$\psi^{*pin}(\alpha, \beta, \varphi) = \frac{3a^2 h \rho \sin \alpha \cos \varphi}{8\pi^2 \mu} \int_0^{\pi} d\alpha_0 \int_{\beta_1}^{\beta_2} h^2(\alpha_0, \beta_0) \sin^3 \alpha_0 ((F_1 + F_2 c h \beta) \cos \alpha_0 - F_2) [\beta_{<} + e^{-(\beta_{>} - \beta_{<})} \cos(\alpha - \alpha_0)] d\beta_0 \quad (50)$$

for $\beta_0 > \beta$, the integration limits are taken from β_1 to β and for $\beta_0 < \beta$, the integration limits are taken from β to β_2 .

Performing α_0 -integration of the last equation we get ,

$$\psi^{*pin}(\alpha, \beta, \varphi) = \frac{3a^2 h \rho \sin \alpha \cos \varphi}{16\pi^2 \mu} \left[\int_{\beta_2}^{\beta} (e^{-(\beta - \beta_0)} I_1) + \beta_0 I_2 d\beta_0 + \int_{\beta}^{\beta_1} (e^{(\beta - \beta_0)} I_1) d\beta_0 \right] \quad (51)$$

where,

$$I_1 = \{ I_{11} \cos \alpha + I_{12} \pi \sin \alpha \} \quad (52a)$$

$$\begin{aligned}
 I_{11} &= J(4/3 - 8ch^2\beta_0) + 6sh^3(\beta_0 - \beta_2)ch\beta_0 + \\
 &\quad \left[-sh^3(\beta_0 - \beta_2)(3ch^2\beta_0 + 1) + J(4ch^2\beta_0 - 2ch\beta_0) \right] \log\left(\frac{ch\beta_0 + 1}{ch\beta_0 - 1}\right) \\
 I_{12} &= J(4ch^3\beta_0 - 4ch^2\beta_0sh\beta_0 - 3ch\beta_0 + sh\beta_0) \\
 &\quad - sh^3(\beta_0 - \beta_2)(-1.5 + 3ch^2\beta_0 + 3ch\beta_0sh\beta_0) \\
 I_2 &= J(-6ch\beta_0) + 4sh^3(\beta_0 - \beta_2) + [sh^3(\beta_0 - \beta_2)(-2ch\beta_0) + J(3ch^2\beta_0 + 1)] \log\left(\frac{ch\beta_0 + 1}{ch\beta_0 - 1}\right), \tag{52b}
 \end{aligned}$$

and

$$J = sh\beta_0(-1 + ch^3(\beta_0 - \beta_2)) + ch\beta_0sh^3(\beta_0 - \beta_2)$$

After performing the β_0 -integrations in Eq.(51) we get

$$\psi^{*pin}(\alpha, \beta, \varphi) = \frac{3a^2h\rho\sin\alpha\cos\varphi}{16\pi^2} [K(\beta)\cos\alpha + M(\beta)\pi\sin\alpha + N(\beta)], \tag{53}$$

where ,

$$K(\beta) = \exp(-\beta) \begin{pmatrix} k1(\beta_2) + .66\beta \\ + k2(\beta) - k2(\beta_2) \end{pmatrix} + \exp(\beta) \begin{pmatrix} k3(\beta_1, \beta_2) - k4(\beta) \\ + k4(\beta_1) \end{pmatrix}, \tag{54}$$

$$+ k5(\beta) + k6(\beta_2) + \exp(-3\beta_2)k7(\beta) + \exp(3\beta_2)k8(\beta)$$

$$M(\beta) = M1(\beta) + \exp(-\beta)M2(\beta_2) + \exp(\beta)M3(\beta_1, \beta_2) \tag{55}$$

$$N(\beta) = N1(\beta) + N2(\beta)\log\left(\frac{1 + e^{-\beta}}{1 - e^{-\beta}}\right) + N3(\beta)\log\left(\frac{e^\beta + 1}{e^\beta - 1}\right). \tag{56}$$

with,

$$k1(\beta_2) = .35 + 1.35\exp(2\beta_2) + 1.04\exp(-2\beta_2) - .08\exp(4\beta_2)$$

$$\begin{aligned}
 k2(\beta) &= -.93\log(1 - e^{2\beta}) + \log\left(\frac{1 + e^\beta}{1 - e^\beta}\right)\left(\frac{3}{8}sh(3\beta_2) + \frac{3}{16}ch(3\beta_2) - \frac{\beta e^{3\beta_2}}{4}\right) \\
 &\quad + \log\left(\frac{ch(\beta) + 1}{sh(\beta)}\right)(.17e^{-3\beta} - .1e^{5\beta} + e^{3\beta_2}(.5\beta + .25e^{-2\beta} + .13e^{-4\beta})) \\
 &\quad + e^{-3\beta_2}(-.25e^{2\beta} + .13e^{4\beta} + .08e^{6\beta} + .08e^{-6\beta} + .06e^{8\beta})
 \end{aligned}$$

$$\begin{aligned}
 k2(\beta_2) &= -.93\log(1 - e^{2\beta_2}) + \log\left(\frac{1 + e^{\beta_2}}{1 - e^{\beta_2}}\right)\left(\frac{3}{8}sh(3\beta_2) + \frac{3}{16}ch(3\beta_2) - \frac{\beta_2 e^{3\beta_2}}{4}\right) \\
 &\quad + \log\left(\frac{ch(\beta_2) + 1}{sh(\beta_2)}\right)(e^{3\beta_2}(.08 + .5\beta_2 - .04e^{2\beta_2} + .38e^{-2\beta_2} - .12e^{-4\beta_2})) \\
 &\quad + e^{-3\beta_2}(.17 + .08e^{-6\beta_2})
 \end{aligned}$$

$$k3(\beta_1, \beta_2) = -.27 - .66\beta_1 - .27e^{-2\beta_1} - .3e^{-4\beta_1} - .66e^{2\beta_1} + e^{-3\beta_2} (.23e^{5\beta_1} + e^{3\beta_1} + .54e^{\beta_1}) + e^{3\beta_2} (.16e^{-7\beta_1} + .58e^{-5\beta_1} + .51e^{-3\beta_1} - 1.2e^{-\beta_1})$$

$$k4(\beta) = .27 \log(1 - e^{2\beta}) + \log\left(\frac{1 + e^\beta}{1 - e^\beta}\right) (e^{-3\beta_2} (.31 + .25\beta) + 3.2ch(3\beta_2) - .06e^{3\beta_2}) + \log\left(\frac{ch(\beta) + 1}{sh(\beta)}\right) (-.17e^{3\beta} - .1e^{-5\beta} + e^{3\beta_2} (.17sh6\beta + .06e^{-2\beta} - .19e^{-4\beta} + .06e^{-8\beta})) + e^{-3\beta_2} (-e^{2\beta} - .06e^{4\beta} - .25\beta)$$

$$k5(\beta) = .08e^{-3\beta} - .58e^{3\beta} + .44e^{-4\beta + 2\beta_2}$$

$$k6(\beta_2) = . - 30e^{5\beta_2} - 2e^{7\beta_2} + 12\beta_2e^{7\beta_2}$$

$$k7(\beta) = -1.2 + 2e^{2\beta} + .56e^{4\beta} + 1.4e^{6\beta} - 12\beta e^{4\beta} + 10e^{8\beta} + 2e^{10\beta}$$

$$k8(\beta) = -1.8 + 21e^{2\beta} - .3e^{-2\beta} + .47e^{-4\beta} + .35e^{-6\beta}$$

$$M1(\beta) = .58e^{-2\beta} - .07e^{-4\beta} - .12e^{5\beta - 3\beta_2} + .94e^{-7\beta + 3\beta_2}$$

$$M2(\beta_2) = .37e^{3\beta_2} + .21e^{-3\beta_2} - .25e^{-\beta_2}$$

$$M3(\beta_1, \beta_2) = e^{-3\beta_2} (-.5 - .5\beta_1) - e^{3\beta_2} (.06e^{-8\beta_1} + .37e^{-2\beta_1}) + .1e^{-5\beta_1} - .17e^{-3\beta_1}$$

$$N1(\beta) = 52.3 + 34ch3\beta_2 + 30e^{-3\beta_2} - sh3(\beta - \beta_2) + 4\beta ch3(\beta - \beta_2) - 19sh(\beta - 3\beta_2) - 4\beta ch(\beta - 3\beta_2) + \beta(-58.7 + 32\beta) + 5.4sh2\beta - 16ch2\beta$$

$$N2(\beta) = -64 + e^{-3\beta_2} (.85 - 11\beta - 6\beta^2) + e^{3\beta_2} (-5.75 - 11\beta + 3\beta^2 - .88e^{-4\beta} - 3.5\beta e^{-4\beta} + 3.5e^{-2\beta} + 7\beta e^{-2\beta})$$

$$N3(\beta) = -2.5sh3\beta + 8\beta ch3\beta - 28sh\beta + 28\beta ch\beta + e^{-3\beta_2} (6\beta^2 - 6e^{2\beta} + 12\beta e^{2\beta} + .25e^{4\beta} - \beta e^{4\beta}) + e^{3\beta_2} (-3\beta^2 + .63e^{-4\beta} + 2.5\beta e^{-4\beta} + 2.5e^{-2\beta} + 5\beta e^{-2\beta})$$

Substituting from Eqs.(53) and (27) into Eq.(44) and then substituting into Eq.(43), the final form of the inertial second-order stream function Ψ_2 is given by

$$\begin{aligned} \psi_2(\zeta, \theta) = & \frac{36h^2c(\alpha_1 + \alpha_2)\sin^2\alpha}{\mu\pi sh^2(\frac{3}{2}\delta)} \{S1(\beta) + S2(\beta)\} \\ & + \frac{3a^2h\rho\sin\alpha\cos\varphi}{16\pi^2\mu} [K(\beta)\cos\alpha + M(\beta)\pi\sin\alpha + N(\beta)] \\ & D_1 \frac{h^3sh^3\beta}{c^3} + D_2 \frac{h}{c}(ch\beta + \cos\alpha) + D_3 \frac{hsh\beta}{c} + D_4 \end{aligned} \tag{57}$$

Applying the boundary conditions, Eq.(26), on the last equation, the functions D_1, \dots, D_4 can be obtained as follows

$$D_1 = \frac{\sigma_1\sigma_2 - \sigma_3\sigma_4}{\sigma_4\sigma_5 - \sigma_1\sigma_6}, \tag{58}$$

$$D_2 = \frac{-1}{\sigma_1} [\sigma_3 + \sigma_5 D_1], \quad (59)$$

$$D_3 = \frac{-1}{g_1(\beta_1) - g_1(\beta_2)} \{ \Gamma(\beta_1) - \Gamma(\beta_2) + D_2 [g_2(\beta_1) - g_2(\beta_2)] + D_1 [g_1^3(\beta_1) - g_1^3(\beta_2)] \} \quad (60)$$

$$D_4 = -[\Gamma(\beta_1) + D_3 g_1(\beta_1) + D_2 g_2(\beta_1) + D_1 g_1^3(\beta_1)], \quad (61)$$

where ,

$$g_1(\beta) = h(\beta)sh\beta$$

$$g_2(\beta) = h(\beta)(ch\beta + \cos \alpha),$$

$$g_3(\beta) = h^2(\beta)(1 - ch\beta \cos \alpha)$$

$$g_4(\alpha) = -2g_1^3(\beta_2) + 2g_1(\beta_1)g_1(\beta_2) + h^2(\beta_2)(1 - ch\beta_2 \cos \alpha)$$

$$\Gamma(\beta) = \frac{36(\alpha_1 + \alpha_2)h^2 c \sin^2 \alpha [S_1(\beta) + S_2(\beta)]}{\mu \pi h^2 (\frac{3}{2} \delta)} + \frac{3a^2 h \rho \cos \varphi \sin \alpha}{16\pi^2 \mu} [K(\beta) \cos \alpha + \pi M(\beta) \sin \alpha + N(\beta)]$$

And the coefficients $\sigma_1, \sigma_2, \dots, \sigma_6$ are expressed by the relations

$$\sigma_1 = -2g_1(\beta_1)h(\beta_1)\cos \alpha - \frac{[g_2(\beta_1) - g_2(\beta_2)]}{[g_1(\beta_1) - g_1(\beta_2)]} g_3(\beta_1)$$

$$\sigma_2 = \Gamma_{,\beta}(\beta_2) + [\Gamma(\beta_1) - \Gamma(\beta_2)] \left(2g_1(\beta_2) - \frac{g_4(\beta_2)}{[g_1(\beta_1) - g_1(\beta_2)]} \right)$$

$$\sigma_3 = \Gamma_{,\beta}(\beta_1) - \frac{[\Gamma(\beta_1) - \Gamma(\beta_2)]g_3(\beta_1)}{[g_1(\beta_1) - g_1(\beta_2)]}$$

$$\sigma_4 = -2g_1(\beta_2)h(\beta_2)\cos \alpha + [g_2(\beta_1) - g_2(\beta_2)] \left(2g_1(\beta_2) - \frac{g_4(\alpha)}{[g_1(\beta_1) - g_1(\beta_2)]} \right)$$

$$\sigma_5 = g_3(\beta_1) \left(2g_1^2(\beta_1) - g_1^2(\beta_2) - g_1(\beta_1)g_1(\beta_2) \right)$$

$$\sigma_6 = -2g_1^4(\beta_2) + 2g_1^3(\beta_1)g_1(\beta_2) + 3g_1^2(\beta_2)h^2(\beta_2)(1 - ch\beta_2 \cos \alpha) - \frac{(g_1^3(\beta_1) - g_1^3(\beta_2))}{g_1(\beta_1) - g_1(\beta_2)}$$

6. Results and discussion

The present work represents a theoretical investigation of the isochoric and isothermal Non-linear flow of a viscoelastic second-order fluid including inertia in the annular region between two eccentric spheres. The results show that the axial component vanishes; i.e. , $W_2(\alpha, \beta) = 0$, while the planar secondary velocity field \underline{U}_2 , with its components u_2 and v_2 , is determined by the stream function Ψ_2 given by Eq. (57). This stream function $\Psi_2(\alpha, \beta)$ is being the solution of the equation of motion, within the retarded motion approximation, and it causes a planar secondary flow field \underline{U}_2 superimposed onto the primary flow which is represented by the first order velocity field $W_1(\alpha, \beta)$, [3].

The general streamlines of the fluid particles due to both of the primary and secondary flows can be visualized by combining the primary motion $W_1(\alpha, \beta)$, which takes a closed path about the rotational axis; i.e. the Z-axis, and the loops of the streamlines mapped onto ρ -Z plane. In view of a specific particle located in a specific circular primary path, when exposed to the additional secondary motion it will draw a closed loop.

Finally, we conclude that the effect of the inertia on flow of a viscoelastic fluid in the annular region between two eccentric spheres is appeared in the equations of motion of second-order fluid and therefore, the solution for the first order does not affected. However, the equation of second order which indicates the presence of secondary flow superimposed onto the primary one, includes inertia term clearly. The inertia second order stream function depend on viscosity and density of the fluid.

The present modified second-order stream function $\psi_2(\alpha, \beta)$, due to the effect of the inertia, which describes the planar secondary flow field has been properly formulated by solving the three-dimensional inhomogeneous biharmonic vector equation. Using Green's function of the Bispherical coordinates, the particular solution of the modified planar second-order equation of motion which including inertia term is obtained and then the complete stream function which describe Non-linear flow of second order fluid are determined.

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